Numerical Solution of Darcy's Law with Memory

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Chapter 1

Introduction

Henry Darcy carried out the following experiment as early as in 1856 in order to study the flow through a porous medium and published his observations.

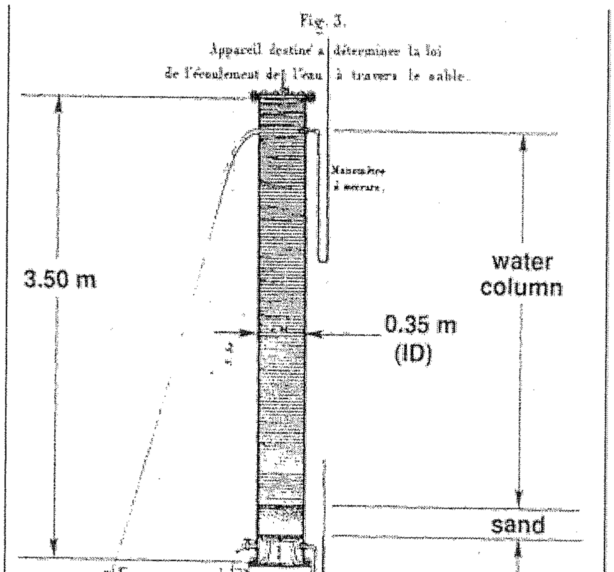

Figure 1.1: Experiment of Henry darcy

He found out that the flow rate u is proportional to the pressure difference ∇p, namely

$$u = \frac{k}{\mu}\nabla p \tag{1.1}$$

where k is the permeability and μ is the viscosity. For several decades developments in this area have taken place in almost independent fields. In civil engineering many papers were published dealing with foundation of flow and transport of fluid through porous media.But there were no rigorous mathematical derivation of the Darcy's law in a porous medium until a new mathematical concept called *homogenization* which deals with partial differential equations having rapidly oscillating coefficients was developed in 1980's. The first mathematical derivations of Darcy's law were done by Keller [1] and Tarter [2]. One might argue that for practical applications mathematical convergence proofs are irrelevant, but this is not at all true. Not only does a mathematical treatment clarify on which assumptions the law is based, but it also allows for development of techniques that can be applied to similar and related problems. This has indeed, been the case. In the last fifteen years a rapid mathematical development has taken place in this area, and now there is a far better understanding of many of the basic phenomena. It was only recently that several non standard models were discovered or justified. One of these is *Darcy's law with memory*

$$\vec{u}(t,x) = \vec{u}_{int}(x) + \frac{1}{\mu}\int_0^t A(t-s)(\vec{f} - \nabla p)(s,x)ds \tag{1.2}$$

where $A(t)$ is the permeability matrix, μ is the viscosity, $\vec{f}$ is an external force and $\vec{u}_{int}$ is the evolution of the initial condition in the medium, which we believe to be a better correction of Darcy's law in instationary flows in a porous medium.

The purpose of this work is to develop an effective numerical algorithm to solve (1.2) and compare it with the classical darcy's law in different situations. In the next section of the chapter we discuss a one dimensional motivative example. Formal derivation and Rigorous mathematical convergence proof of Darcy's law with memory is presented in Chapter 3 and Chapter 4 respectively.

It is clear that the major difficulty one has to overcome in solving equation(1.2) is the parameter dependent integral. Discretization of (1.2) leads to a system of partial differential equations which can be solved recursively as time evolves, but the inhomogeneity of such an equation at a given time level depends on the summation of all previously calculated solutions, therefore this process cannot be continued since the accumulation of errors creates instabilities in the numerical methods. This observation compel us to develop an alternative approach in order to get rid of the parameter dependent integral.

Using semigroup theory we prove in Chapter 4 that the time dependent permeability matrix $A(t)$ can be written in a special form and it gives us a key idea to overcome the above mentioned difficulty. Chapter 5 is devoted to develop the necessary numerical schemes.

First of all we introduce basic concepts of finite difference methods and then a semi implicit numerical scheme to solve the incompressible Stokes equations

$$\begin{aligned}
&\frac{\partial \vec{u}}{\partial t} + \nabla p - \mu \Delta \vec{u} = \vec{f} \\
&div\ \vec{u} = 0 \\
&+ \text{ initial and boundary conditions}
\end{aligned}$$

which is called the *auxiliary* problem in Chapter 3. By solving the auxiliary problem we determine the permeability matrix $A(t)$ in its special form. This allows us to write our equation (1.2) as a first order partial differential equation which is easier to handle and has no parameter dependent integral. Finally we present numerical results and our conclusions.

1.1 One dimensional model for Resin Transfer Molding(RTM) process using Darcy's law with memory

According to Kai [41], it is important to study one dimensional models for RTM processes in oder to understand interaction between various parameters in the process. He studied the process using the classical Darcy's law and obtained implicit analytical expressions for important quantities such as wet length,filling time etc. Purpose of studying this example is to investigate whether the one dimensional model for RTM processes using Darcy's law with memory gives better results than that of classical Darcy's law.

We assume that we are given the RTM process parameters in Table (1.1) as the data of the flow problem to be solved. Our objective is to calculate the quantities listed in the Table (1.2)